Next consider the six Borda outcomes that have only one tie, which are items 7–12. There are 30 sets of pairwise outcomes that are consistent with these Borda results (5 for each one). Considering Borda ordering 7, we find that the following are the legal words.

$$\begin{array}{llll} 1.\ x \succ_P y, & x \succ_P z, & y \approx_M z, & x \succ_{BC} y \approx_{BC} z \\ 2.\ x \succ_P y, & x \succ_P z, & y \succ_P z, & x \succ_{BC} y \approx_{BC} z \\ 3.\ x \succ_P y, & z \succ_P x, & y \succ_P z, & x \succ_{BC} y \approx_{BC} z \\ 4.\ x \succ_P y, & x \approx_M z, & y \succ_P z, & x \succ_{BC} y \approx_{BC} z \\ 5.\ x \approx_M y & x \succ_P z, & y \succ_P z, & x \succ_{BC} y \approx_{BC} z \end{array} \tag{3.8}$$

With five more legal words for each of the six Borda outcomes, we have thus added $5 \cdot 6 = 30$ valid words. The proof of this result is left as an exercise for the reader (but it is very informative, and so a complete answer is included at the end of the book).

Finally, consider Borda ordering 13. If the Borda count yields total social indifference, $x \approx_{BC} y \approx_{BC} z$, only three sets of pairwise matchings are possible:

$$\begin{array}{llll} 1.\ x \succ_P y, & y \succ_P z, & z \succ_P x \\ 2.\ y \succ_P x, & z \succ_P y, & x \succ_P z \\ 3.\ x \approx_P y, & y \approx_P z, & z \approx_P x \end{array} \tag{3.9}$$

The first two are the cyclical pairwise comparisons, and the final is generalized indifference. With those three legal words, our total is now 135.

3.6.4 The Schulze Method

If you search the Internet, this one is often found under the unseemly titles "Clone-proof Schwartz Sequential Dropping" or the "Beatpath Method." We have named it for its originator, Markus Schulze, who has been an active proponent of the method since 1997 and has shown that it has many desirable qualities (Schulze, 2003). This method evolved in a sequence of e-mail and Internet postings by a group of enthusiasts who sought to develop workable voting methods that can actually be put to use in "real life." Schulze notes that this method is used in organizations that have an aggregate membership of more than 1700 and that it is the most widely used of all of the Condorcet round robin methods. People who use the Linux operating system might be interested to know that the Schulze method is used in decision making in the Debian Linux project, one of the most widely disseminated Linux distributions.

Of all the methods we have considered so far, this is the most difficult to understand and interpret at face value. To justify it, the proponents do not rely on intuition.

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Rather, they rely on the mathematical fact that this procedure meets many of the criteria against which voting procedures are judged. If there is a Condorcet winner, this method selects it as the winner. The Schulze method satisfies many of the other most important priorities, including undifferentiatedness (same as neutrality), anonymity, monotonicity, independence of clones (cloneproofness), and several others.

Schulze's method also satisfies a property known as Smith-IIA. Recall the Smith criterion, which held that the winner should be a member of the Smith set. Smith-IIA adds the additional stipulation that if a candidate is added in an election, it can only affect the election outcome if it is in the Smith set. If it is not in the Smith set, then it is an irrelevant alternative (so far as the choice among alternatives in the Smith set is concerned) and it should not matter.

The voters submit rankings of as many candidates as they wish, and the ones they do not rank are assumed to be interchangeable and less desirable than the ones they do rank. There are two mathematically equivalent descriptions of the Schulze method for picking the winner in a competitive election. These methods use the idea of "transitive defeat." A "transitive defeat" (also known as a chain) allows x to assert itself against y through a chain of comparisons, like $x \succ_p w$ and $w \succ_p y$. Even if y defeats x in a head-to-head comparison, if a chain from x to y exists, then x can "transitively defeat" y. You might think of the chain as a sort of "self-defense" or "political countermeasure" for electoral candidates. The chains can be quite long, and to save space we might as well write $x \succ_p w \succ_p z \succ_p y$ to mean that x transitively defeats y with w and z in the middle of the chain.

As the old saying goes, a chain is only as strong as its weakest link. The strength of a chain from x to y is measured by the smallest margin of victory between any two alternatives in the chain. The symbol $P[x, y]$ represents the strength of the strongest chain from x to y. If there are two chains leading from x to y, we measure the strength of each one (i.e., find the smallest margin of victory), and $P[x, y]$ is the largest value. If there is no chain from x to y, then the strength is $P[x, y]=0$.

The "beatpath method" is the one that Schulze uses as the definition of the social ranking process. The aim is to find an alternative with the strongest chains through which it can defeat all other alternatives. If the chain from x to y is stronger than the chain from y to x, $P[x, y] > P[y, x]$, then x disqualifies y from further consideration. If we collect up all the alternatives that are not disqualified, then we have a set of "potential winners." If we are lucky enough to have only one candidate left, then it is the beatpath winner. If we have several left, then a tie-breaking procedure is needed. Quite frankly, the emphasis in this method is to narrow the alternatives down to this set of unbeaten candidates, and the tie-breaking procedure is something of an afterthought.

To bring this down to earth, consider the application of this method to choosing a national football champion after gathering votes from sports writers and coaches. The voters rank three undefeated teams, Auburn (AU), the University of Southern California (USC), and the University of Oklahoma (OU). In Figure 3.4, the arrow from USC to OU indicates that USC is preferred to OU by 44 of the voters. Here, we are using the number of voters as the indicator of the strength of the defeat (we

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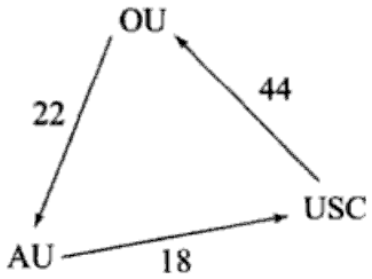

Figure 3.4 *Beatpath for college football.*

could use the margin instead). Note there is a cycle, where OU is ranked higher than AU (strength = 22) and AU is preferred to USC (strength = 18). This example is particularly simple because there is only one path from each alternative to each of the others. The beatpaths are as follows:

1. P [*OU*, *AU*] = 22
2. P [*OU*, *USC*] = 18
3. P [*AU*, *USC*] = 18
4. P [*AU*, *OU*] = 18
5. P [*USC*, *OU*] = 44
6. P [*USC*, *AU*] = 22

Because P [*USC*, *OU*] > P [*OU*, *USC*] and P [*USC*, *AU*] > P [*AU*, *USC*], *USC* is the beatpath winner.

The implications of the beatpath calculations are difficult to grasp for many people. Why does the procedure satisfy so many of the desirable qualities? How can we make sense of it? Shortly after Schulze proposed his method in 1997, it was shown that it is equivalent to a method in which we sequentially measure the strength of the candidates and then remove the weak ones, and then reevaluate. The sequential evaluation process is organized around an enhancement of the Smith set, which is called the Schwartz set. This approach is often called clone-proof Schwartz sequential dropping (CSSD) because we sequentially recalculate the Schwartz set after penalizing the weak candidates.

To be a member of the *Schwartz set*, an alternative must be either undefeated in pairwise competition (it may defeat *or tie* each other alternative) or able to transitively defeat every other alternative that can transitively defeat it. To be in the Schwartz set, a candidate must have "electoral self-defense" against each of the other candidates. More formally, x is in the Schwartz set if for any y such that P [y, x] > 0, then P [x, y] > 0. Even if x transitively defeats y, y can be in the Schwartz set if there is a chain of plurality decisions reaching from y to x. An alternative y is rejected from the Schwartz set if y cannot transitively defeat another alternative that can transitively defeat y. If there are no pairwise tie votes, the Schwartz set equals the Smith set. If there are ties, then the Schwartz set can be much smaller than the Smith set. In Figure 3.5, ties are represented by dotted lines that connect alternatives. The ties

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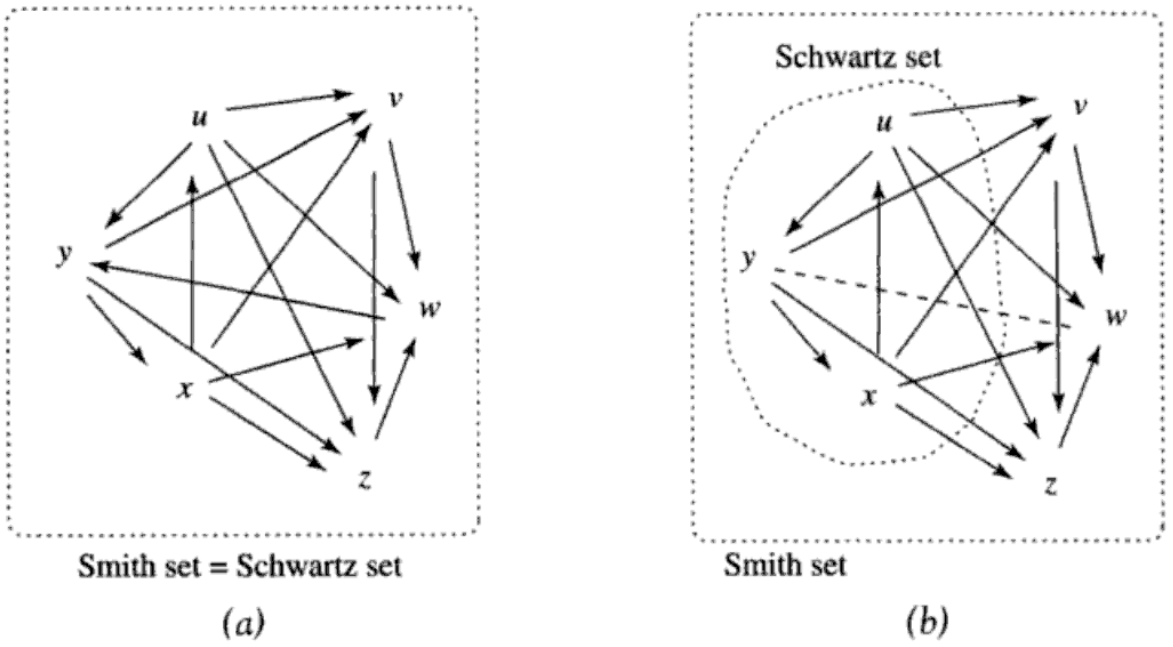

Figure 3.5 *Schwartz sequential dropping. (a) Smith set = Schwartz set. (b) One tie shrinks the Schwartz set. (c) Schwartz set shrinks even further.*

make a difference because the transitive defeat concept is based on pairwise victories and ties are treated the same as losses. In other words, ties break chains.

The sequential approach can be thought of as a way to find the strongest chains. The Schwartz sequential dropping version of the Schulze procedure is the one that is described in the Constitution of the Debian software organization *http://www.debian.org/vote/2003/vote_0002*. In Figure 3.5, we illustrate the sequential narrowing of the Schwartz set that results when ties (dotted lines) are inserted.

Schwartz sequential dropping proceeds iteratively, repeatedly narrowing down the alternatives. It is easy to make this sound more complicated than it really is, but we will try not to.

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Step 1. Collect the ranking information from the voters. Voters are allowed to rank all the alternatives or just some of them. They are allowed to register their indifference between alternatives. From the ballots, we can calculate the pairwise outcomes for all alternatives, either $x \succ_P y$ or $x \approx_P y$ or $y \succ_P x$.

Now the decision-making starts.

Step 2. Eliminate rejects. Begin by simplifying the problem by isolating "rejects," alternatives that are losers in the round robin tournament. These losers are the ones left out of the Schwartz set. Recall that y is rejected if it is transitively defeated by some other proposal and there is no chain that leads from y to that other alternative. The rejects are eliminated from consideration, permanently.

Step 3. Evaluate the remaining contenders (members of the Schwartz set).

A. If there is only one remaining contender, then it is the winner.

B. If there is more than one contender, check the pairwise comparisons among all the contenders. If all pairs are tied, it means, in some sense, that we have found all the alternatives that are "equally good." If all are tied, proceed to the tie-breaking procedure in step 4.

C. If all contenders are not tied pairwise, then we need to make some changes so we can reject more alternatives. Here is where the "sequential dropping" kicks in. Among all pairs of contenders, find the "weakest pairwise defeat." (To us, a name like "least impressive victory" would be more fitting.) The idea is to spot the least worthy pairwise victor and demote it. How to spot the least impressive winner? Look for the alternative that wins with the smallest number of votes.[3] When the weakest pairwise defeat is found, that victory can be erased by changing it to a tie in the data. That is, we change the result in the round robin data for the weakest defeat from $\succ_P$ to $\approx_P$. (Caution: Contrary to what some websites say, neither the weakest defeat nor the tied alternatives are not dropped literally. The weakest defeat is converted to a tie and the Schwartz set is recalculated.) If there are several alternatives that are equally weak winners—many weakest defeats exist—then choose the one with the most votes for the loser and declare that one to be the weakest defeat. If there are several pairs tied in both the number of votes for the winner and the loser, treat them all in the same way: Change the pairwise comparison to a tie. This generates new round robin data.

Go back to step 2 and repeat the procedure. Use the new round robin data, of course. The rejects that were eliminated in previous stages are never readmitted. Because we have inserted some ties into the results, the members of the Schwartz set

[3]Some versions of this procedure use the margin of victory rather than the absolute number of votes received by the winner. Schulze recommends the total number of votes because he believes it gives voters a positive incentive to report their rankings as completely as possible. Otherwise, they might have an incentive to report just the rankings of the top one or two alternatives to "game" the election system.

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will change: Some contenders will become rejects and then they are dropped. The procedure will repeatedly loop between steps 2 and 3 until the condition described in step 3B is obtained.

Step 4. If we arrived at this stage, it means we have narrowed down the alternatives to the most appealing set of more or less equally good outcomes. Schulze calls these "potential winners." A winner is selected by a tie-breaking rule.

The tie-breaking rule that is suggested by Schulze is, more or less, going to rank the tied outcomes by listening to the opinions of randomly drawn voters. Suppose the tied candidates are $\{w, x, y, z\}$. Randomly draw a ballot and copy down the preferences of the voter. If that ballot states a ranking for all the alternatives, then that is the end of the procedure. The group result will match that ballot. If the ballot is only partially filled in, for example, it states only $x \succ_i y$, then we take that information and mark down $x \succ_{\text{Schulze}} y$ (it is "set in stone," unalterable) and we draw another ballot. If that ballot has preference information about candidates that are not already sorted by $\succ_{\text{Schulze}}$, such as w and y, then use that information $w \succ_{\text{Schulze}} y$. We draw ballots at random until every pairwise comparison is filled in, and we *never* change any of the Schulze comparisons we have already completed. If we happen to go through all ballots without ranking all the alternatives, then the final ranking is assigned at random.

The Schulze method can be applied to any of the voting examples we have considered. Of course, if the preferences imply the existence of a Condorcet winner, then that alternative is chosen and the solution is simple. To find a more interesting case, we need to select an example in which there is a voting cycle. Recall the Borda paradox considered in section 3.4.2. In Figures 3.6 and 3.7, we have used the preferences from Tables 3.3 and 3.4 with the Schwartz sequential dropping approach.

In Figure 3.6, we note that the original preferences allow a majority rule cycle across all of the alternatives, so the first-stage Schwartz set includes all the alternatives. There are two comparisons tied for the weakest defeat, each with the support of four voters, so in Figure 3.6b, those arrows are changed to dotted lines representing ties. After those ties are taken into account, the Schwartz set is recalculated, and it includes only w.

Recall that alternative z is exposed as a fraud in that example. The effect of dropping z to the bottom of the voter preference orderings appears in Figure 3.7. Note that the initial Schwartz set now includes three alternatives, $\{w, x, y\}$. Alternative z is excluded (it is a Condorcet loser). The weakest defeat is between w and y. After that is changed to a tie, then w is the only potential winner remaining. The same alternative, w, wins with the Schulze method in both cases.

The Schulze method has the apparent strength of solving the paradox in the Borda count: w wins in both cases. The Schulze method is thus more stable and, in some sense, more logical. The fact that the Borda method chose y in the original example (Table 3.3) and Schulze chooses w deserves mention. From a beatpath point of view, it appears that we did not correctly understand the Borda problem in section 3.4.2. We thought the paradox was that y was turned from a winner into a loser when the fortunes of z changed. Now it appears that the original Borda winner, y, was

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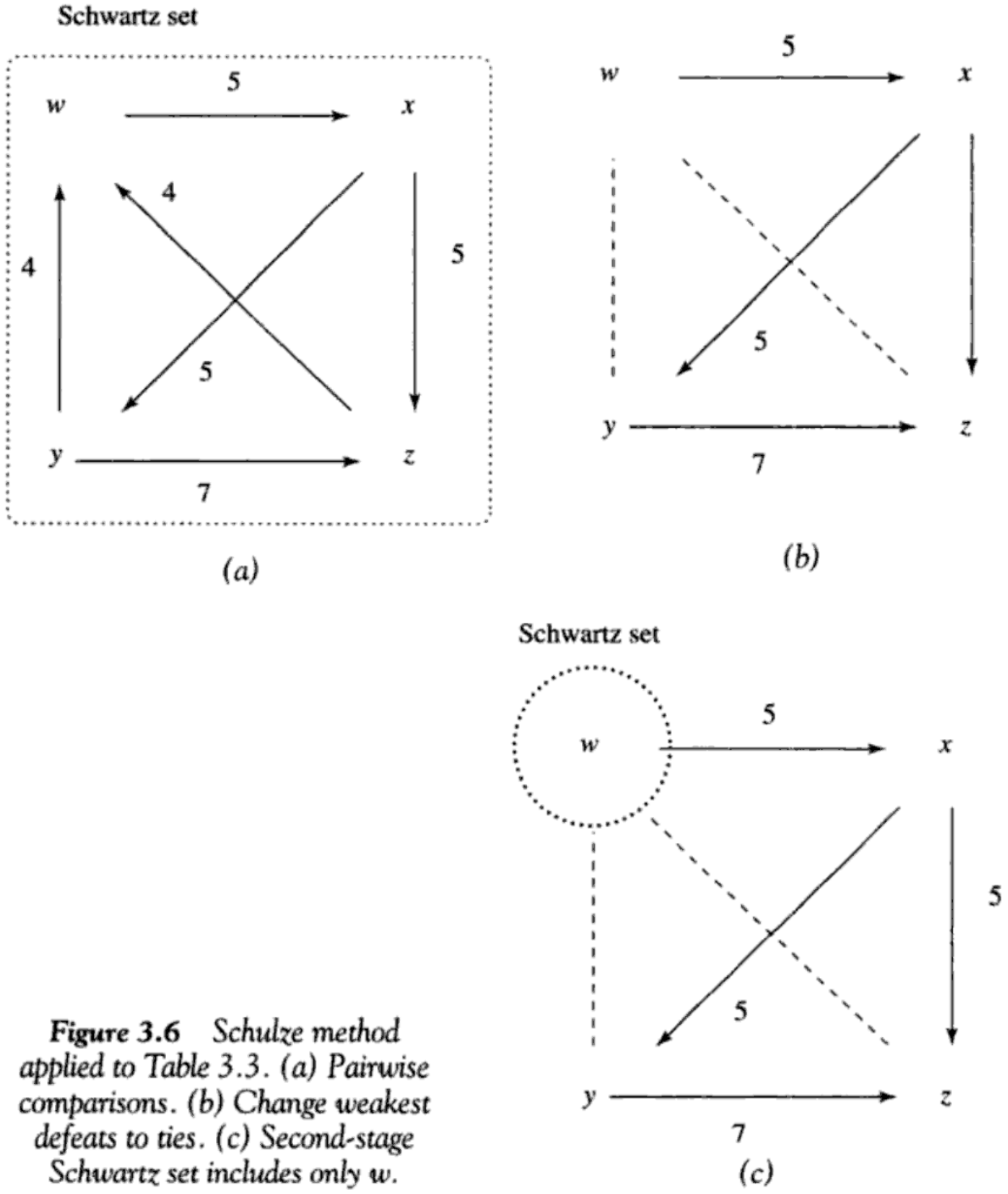

Figure 3.6 *Schulze method applied to Table 3.3. (a) Pairwise comparisons. (b) Change weakest defeats to ties. (c) Second-stage Schwartz set includes only w.*

"wrong." Borda chose the wrong winner because it allowed z to play the role of a spoiler, somehow distracting voters from their purpose, thus allowing y to win. When z is moved to the bottom in Table 3.4, its role (its influence as an irrelevant alternative) is eliminated and the "correct alternative" w wins.

The Schulze method leads to a full transitive ordering of all the candidates. The beatpath values $P[x, y]$ can be used to sort alternatives from top to bottom. As a result, the method could be used to determine the winner of the Division 1A College football championship on the basis of the rankings submitted by coaches, writers, or computers.

The reader might be wondering, "if this method is so great, why isn't everyone using it?" If you ask its proponents, we expect they will say "because everyone is not

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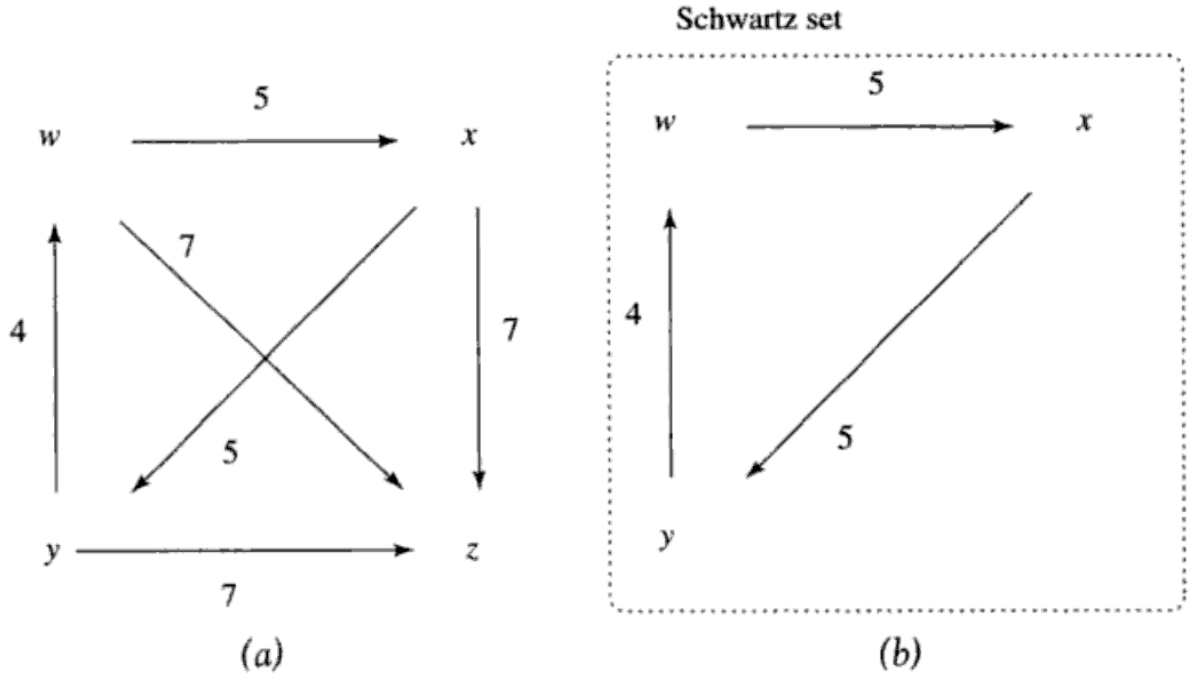

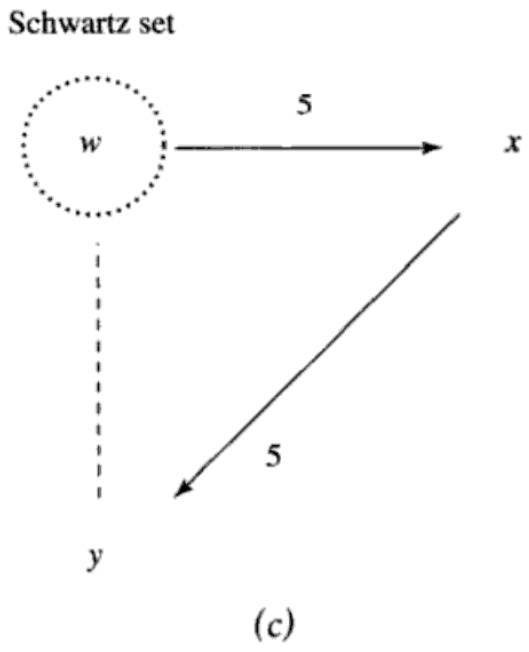

Figure 3.7 *Schulze method applied to Table 3.4. (a) Pairwise comparisons. (b) First-stage Schwartz set excludes z. (c) Second-stage Schwartz set includes only w.*

very smart!" Ordinary people—not academics—are not particularly interested in election procedures, much less complicated ones. As a result, the most widely used argument against the method seems to be that it is too complicated. To academics, this is not a very good argument, but it carries a lot of weight with real-life policymakers. In response, the Schulze proponents will point out that, from a voter's point of view, it is no more complicated than a Borda count. All the voters need to do is submit rankings of some alternatives. The counting process is more complex, of course, but the effort is justified. The proponents can list a long list of favorable qualities that we have already mentioned, plus some more.

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The beatpath approach is not completely free from the anomalies that infect the other voting mechanisms. Schulze draws our attention to the fact that his method violates the *participation criterion*. The participation criterion says that if an election has been decided and then you add voters who prefer the winner to another alternative, then adding those new voters must not result in the victory of that other (less favored) alternative. More formally, the winner is x, hence socially $x \succ_{\text{Schulze}} y$. Add more voters such that $x \succ_i y$. Adding those voters should not change the social result to $y \succ_{\text{Schulze}} x$. The criterion does not require that x must still win, only that adding voters who do not prefer y should not make y the winner. Please note that the participation criterion is different from the monotonicity principle. Monotonicity states that *among existing voters*, a change in voter preferences that favors x against y must not hurt x's position in the final decision. Participation concerns the addition of new voters. Schulze is a monotonic procedure, but examples can be found to show that it violates the participation criterion.

Schulze's example of the violation of the participation criterion is presented in an example that has six candidates, $\{u, v, w, x, y, z\}$ (Table 3.11). The winner is u when the electorate includes the 60 voters from groups 1–12. The violation of the participation principle is shown when group 13, which has three members, is added to the electorate. For all the members of group 13, note that u is the most preferred alternative and x is the least desirable. One would expect that their favorite, the winner u, would remain in good position. Unfortunately, the Schulze rule applied to this 63 voter electorate declares x to be the winner. After the sequential dropping process, u is the only contender (the only member of the final Schwartz set), and no use of the "tie breaker" is required.

Among the collection of peculiar election outcomes that we have been accumulating, this is surely one of the most bizarre. To investigate it more carefully, consider the graphs that illustrate the sequential dropping process in Figure 3.8. On the

Group	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of voters	4	2	4	2	2	2	4	12	8	10	6	4	3
Preferences													
Favorite	u	u	u	u	v	w	w	x	y	z	z	z	u
Second	v	v	y	y	z	x	x	y	w	u	u	y	y
Third	w	z	v	z	u	v	v	w	x	v	v	x	z
Fourth	x	x	z	v	w	y	z	u	v	w	x	v	w
Fifth	y	y	w	w	x	z	y	v	z	x	y	w	v
Sixth	z	w	x	x	y	u	u	z	u	y	w	u	x

Table 3.11 The Schulze method

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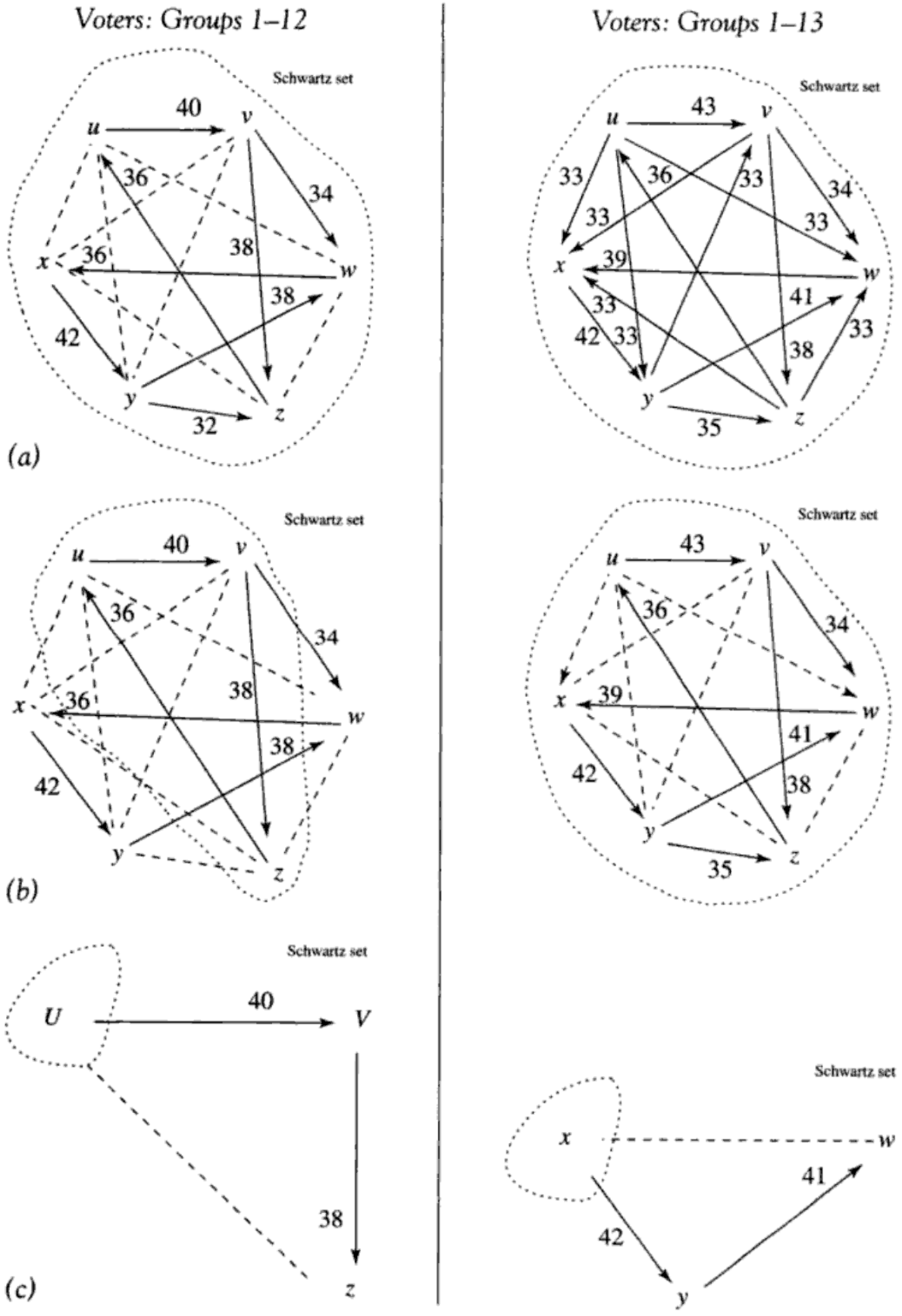

Figure 3.8 *Schulze method with preferences in Table 3.11. (a) Pairwise comparisons. (b) Change the weakest defeats to ties. (c) Final stage.*

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left, the pairwise data for the 12-group problem is illustrated. Note there are many ties in this example. The weakest defeat is the step $y \succ_P z$ which is supported by 32 voters. After that is changed to a tie, the Schwartz set shrinks to $\{u, v, w\}$, and from there it is easy to drop the weakest defeat and see that u wins. On the right side of the figure, note that when group 13 is added, there are no pairwise ties. When the weakest defeats are changed to ties, then the pattern of ties and pairwise comparisons in the initial results on with 12 groups reappears. The ties are the same, but the margins of victory are different. With 13 groups, the weakest defeat is now $v \succ_P w$. After a tie is drawn between v and w, the Schwartz set shrinks to $\{x, w, y\}$, and from there the result is easy to see: x wins. Why did u lose? When the path from v to w was removed, then u lost its chains of self-defense. The alternatives x, y, and z can all transitively defeat u, and there is no path leading from u back to them if $v \approx_P w$.

This is a variant of the problem of independence from irrelevant alternatives. What is driving the problem here is not the relative evaluation of u and x but rather the careful placement of u, v, w, and z in the orderings of group 13 voters. The careful placement ends up changing the magnitude of pairwise victories, thus impacting the order in which the sequential dropping eliminates candidates. If you think of the sequential dropping process as a series of games in a tournament, then, we have simply found another example of the way in which the ordering of the matches affects the outcome.

EXERCISES 3.6

1. The following table describes the tastes of three voters. Conduct a round robin tournament and use the win–loss records to try to decide which candidate should win the tournament.

	Voters		
	1	2	3
Ranking			
First	x	y	z
Second	y	z	x
Third	z	x	y

2. Consider the table in the previous question. Describe a couple of changes you might make in voter preferences (either by changing voters or adding new ones) if you want z to become the winner in the round robin.

3. For the following preference profile, conduct a round robin tournament and calculate the aggregated pairwise vote totals for the candidates. Confirm that the results are the same as the Borda count.

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	Voters		
Groups	1	2	3
Number of members	22	12	31
Ranking			
First	x	y	z
Second	y	x	y
Third	z	z	x

4. Consider the preferences used in the previous problem. Create a graph of the type shown in Figure 3.4. Use that graph to figure out which candidate would win with the Schulze procedure.

5. Consider the following preference profile. Create a beatpath graph of the type shown in Figure 3.4.

	Voters					
Groups	1	2	3	4	5	6
Number of members	200	150	150	300	100	100
Ranking						
First	z	y	x	w	y	x
Second	x	z	y	z	x	y
Third	y	x	z	x	w	z
Fourth	w	w	w	y	z	w

6. Use the graph you created in the previous question to figure out which alternatives are in the Schwartz set. If the Schwartz set contains more than one element, proceed with the sequential dropping process (change the weakest defeats to ties and recalculate the Schwartz set).

7. This is a challenging problem, one which we refer to in the feature box in Section 3.6.3. Show that if the Borda count (or, equivalently, the aggregated pairwise vote) outcome is $x \succ_{AP} y \approx_{AP} z$, then the only pairwise outcomes that are consistent with that outcome are the ones listed in equation (3.8).

3.7 Single Vote Systems: Cousins of Plurality and Majority Rule

We began this chapter with the plurality problem. A plurality election with 20 or 30 candidates might award victory to a very unpopular candidate. The poster children for the problem might be the Democratic and Republican candidates in Minnesota's gubernatorial election who were edged out by Jesse Ventura. In the words of mathematician Keith Devlin, Jesse Ventura "won not because the majority of the voters

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